**Twin-edged polyhedra from links and knots. Draft by Stephen Nurse June 2023**

**Abstract**

This paper presents ways of making polyhedra based on a cube documented by Marleen Hartogg. Models are made using paper cutouts, or basket cane trained over jigs and glued. 18 styles of Platonic solids and 13 styles of tessellation are identified.

Basket cane models of the platonic solids exist in configurations which vary the strand count of the model. For example, one style of twin edged tetrahedron has 64 configurations and can be made from 1, 2, 3 or 4 strands of cane.

There is considerable scope for these models to be made and enjoyed in classrooms where doing maths can motivate both students and teachers.
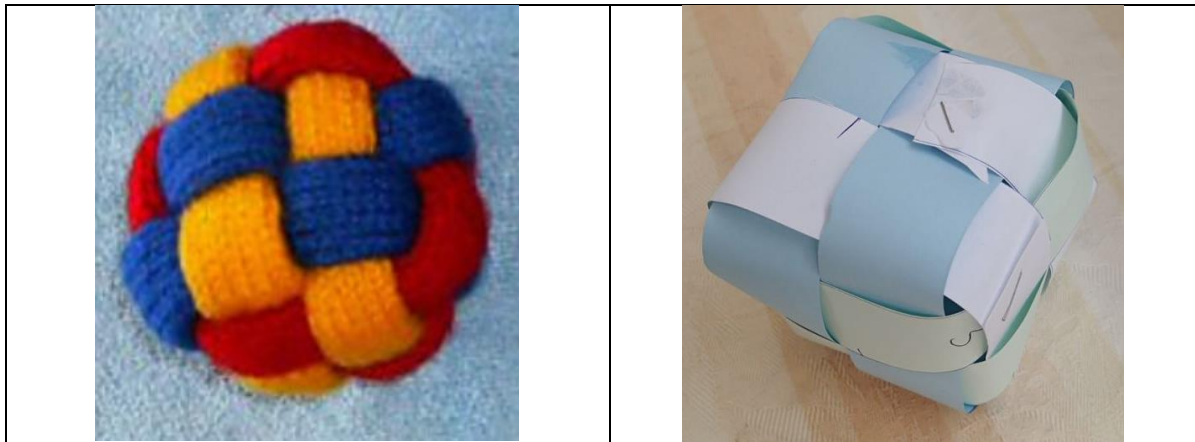
Figure 1: Marleen Hartog's cubic knitted ball (Hartog)

Figure 2: Equivalent model from paper and staples

**Introduction**

In May 2022, a friend Christine suggested a craft idea for my wife: the project was the braided knitted ball by Marleen Hartog (figure 1). Marleen's pattern included a practice paper version which I made (figure 2). Then I wondered: could this form be applied to other polyhedra? If Marleen's ball was a cube, what would a tetrahedron, octahedron, dodecahedron and icosahedron look like?
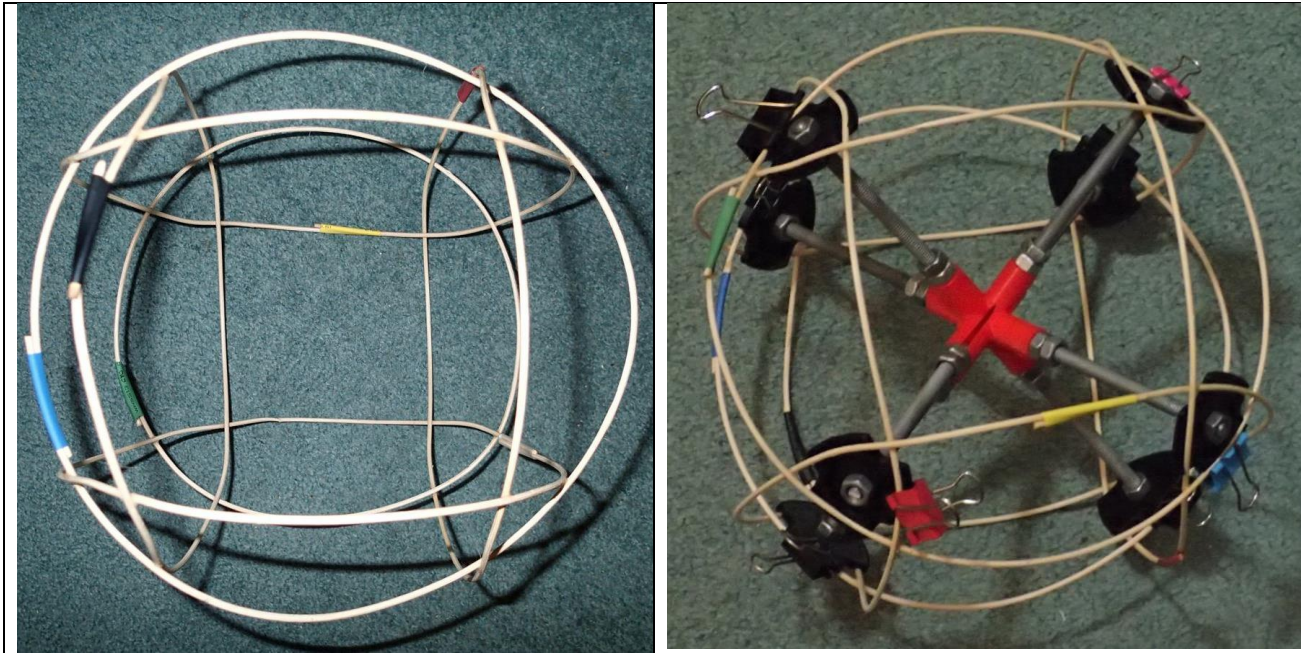
Figure 3: Equivalent of figure 2 from basket cane, electrical heat shrink, glue

Figure 4: Cube from figure 3 in 3d printed jigs during construction

Before answering this, a tractable form of the cube was needed, that is a cube with translatable geometry. This is shown in figure 3. This basket cane has the topology of figure 2 and traces edges of a cube with chamfered edges and truncated vertices. It is a template for similar tetrahedra, octahedra, dodecahedra and icosahedra.
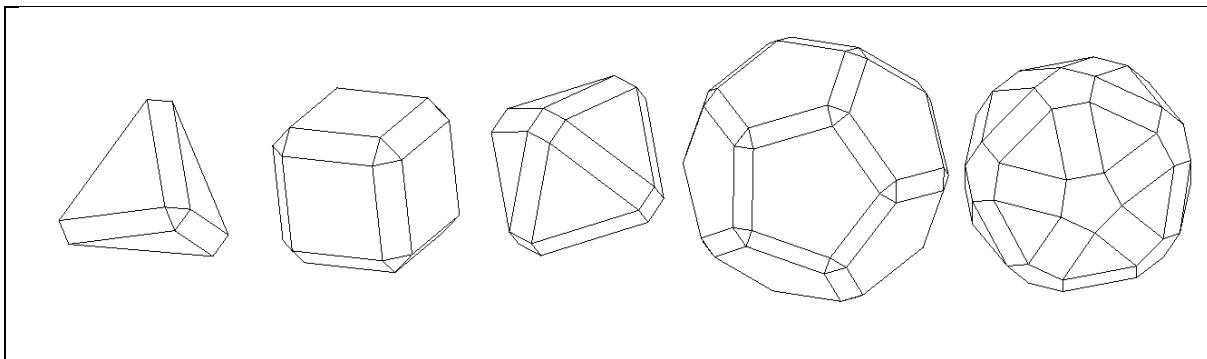


Figure 5: Platonic solids chamfered, then truncated: tetrahedron, cube, octahedron, dodecahedron, icosahedron.

As shown in figure 5, the Platonic solids all have versions which are chamfered, then truncated. It seems logical that equivalents of figure 3 exist for all of them, ie versions with twin edges, and a hoop parallel to each side.

**Tetrahedron and octahedron**

Initial analysis of these forms was on the tetrahedron, and later I considered other shapes.

Figure 6: Stereographic projection sketch of tetrahedron from links, with links representing sides.

Figure 7: Basket cane version of figure 6 held with glue and electrical heat shrink.

Figure 8: Figure 7 tetrahedron was made with clips and 3d printed jigs.

The tetrahedron is the Platonic solid with the least number of faces, edges and corners, so seemed the easiest to sketch and make. I sketched a tetrahedron in stereographic projection, then used jigs to make the interlocking links (Figures 6, 7, 8).
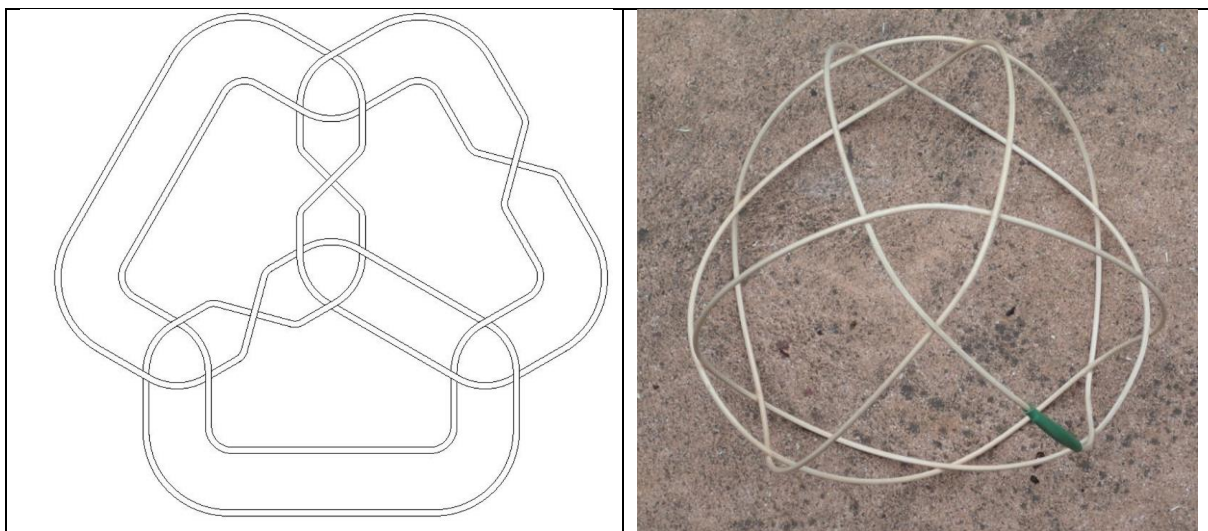


Figure 9: Stereographic projection style sketch of tetrahedron from single strand.

Figure 10: 3d version of figure 5 from basket cane, glue, electrical heat shrink.

Later I sketched variations on the plain stereographic projection, and made a configuration of figure 7 from a single strand. This new shape (Figures 9 and 10) turns the four links into a single knot forming a closed loop or Euler circuit. The Euler circuit depends on half twists in edges acting as switches, transferring strands between links.
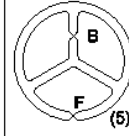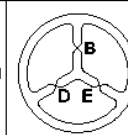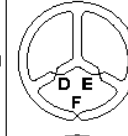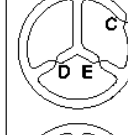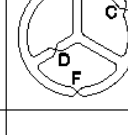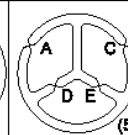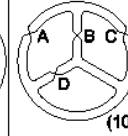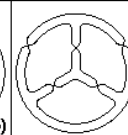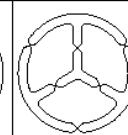
| # of switches in sites ABCDEF | | | | | | |
|---|---|---|---|---|---|---|
| **0** | **1** | **2** | **3** | **4** | **5** | **6** |

| # of links | | 0 | 1 | 2 | 3 | 4 | 5 | totals | |
|---|---|---|---|---|---|---|---|---|---|
| | 4 | 1 | | | | | | | 1 |
| | 3 | | 6 | | | | 1 | | 7 |
| | 2 | | | 15 | 4 | 5 | 6 | | 30 |
| | 1 | | | | 16 | 10 | | | 26 |

Figure 11: Tetrahedron sketches showing configurations of twists in edges. It shows the number of links per configuration, and counts the configurations with 4, 3, 2 and 1 link.

There are six sites for switches in this double-edged tetrahedron, so $2^6$ configurations. Using sketches (figure 11), I counted switching in tetrahedra (see figures 6 to 10), finding:

1   configuration for shapes with 0 or 6 switches,

6   configurations for shapes with 1 or 5 switches

15 configurations for shapes with 2 or 4 switches

20 configurations for shapes with 3 switches.

These results can be calculated through the combination function C(6, k) where k is the number of switches. Harder to predict is the number of links resulting from different switch configurations.

The links in each configuration were counted. Many were similar once symmetry is considered (figure 11). This style of tetrahedron has 64 edge configurations. When 4 links corresponding to faces.

A single strand octahedron is shown in figures 12, 13). Note that vertices become complex when more than 3 edges meet at corners.

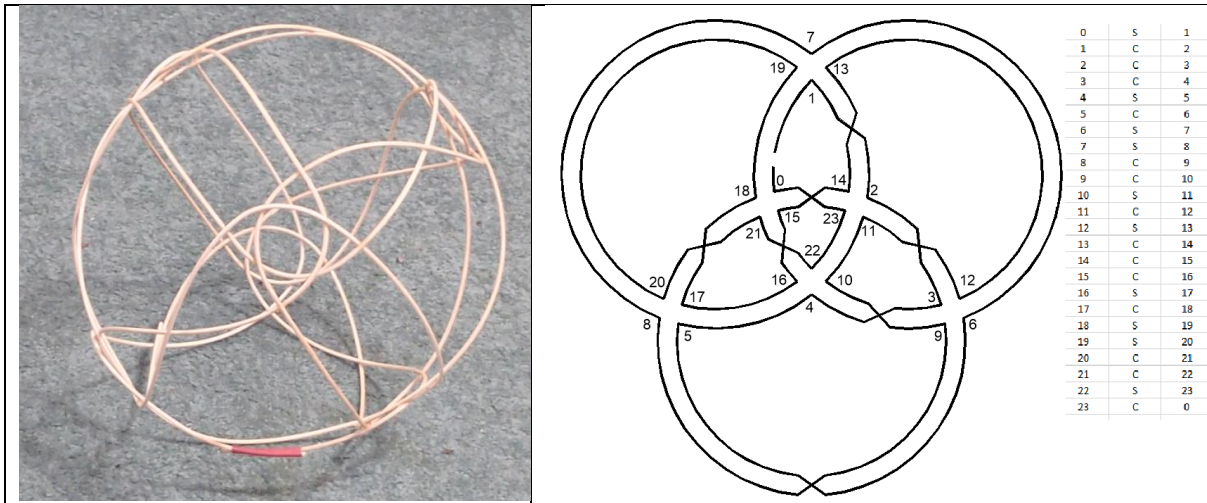| 0 | S | 1 |
| 1 | C | 2 |
| 2 | C | 3 |
| 3 | C | 4 |
| 4 | S | 5 |
| 5 | C | 6 |
| 6 | S | 7 |
| 7 | S | 8 |
| 8 | C | 9 |
| 9 | C | 10 |
| 10 | S | 11 |
| 11 | C | 12 |
| 12 | S | 13 |
| 13 | C | 14 |
| 14 | C | 15 |
| 15 | C | 16 |
| 16 | S | 17 |
| 17 | C | 18 |
| 18 | S | 19 |
| 19 | S | 20 |
| 20 | C | 21 |
| 21 | C | 22 |
| 22 | S | 23 |
| 23 | C | 0 |

Figure 12: Twin edge octahedron from single strand.

Figure 13: Sketch for figure 12 is based on the stereographic projection of an octahedron.
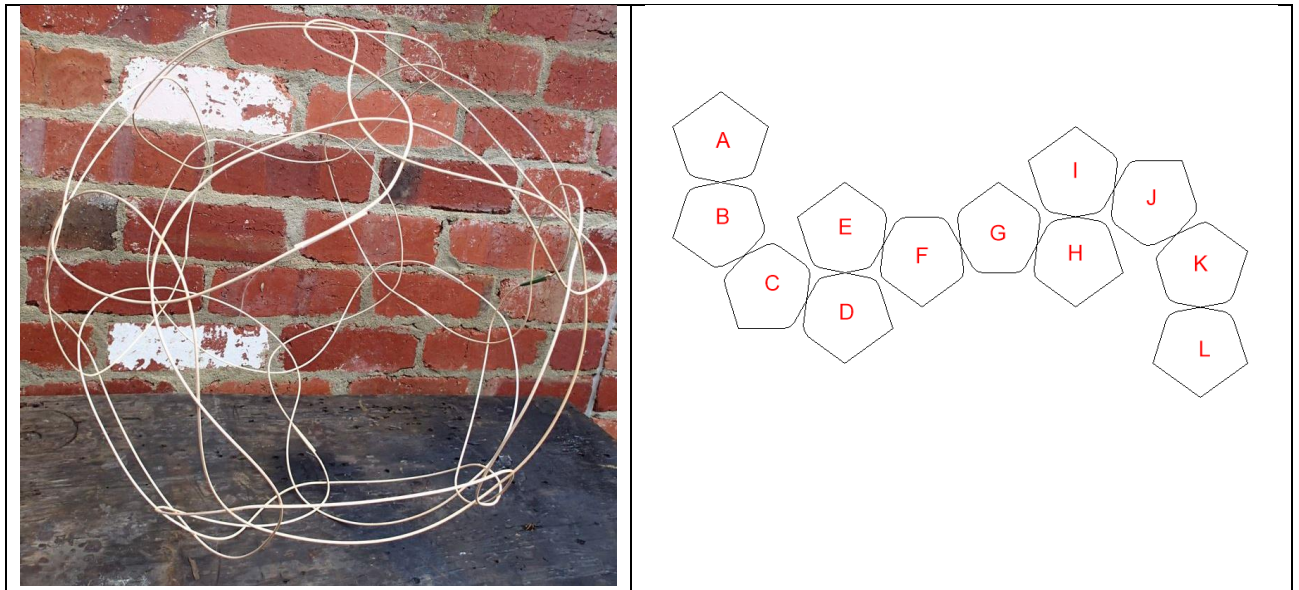
**Dodecahedron and Icosahedron**



Figure 14: Dodecahedron knot from single basket cane strand.

Figure 15: Planning diagram for dodecahedron knot.  (Nurse 2023)

The largest twin edge polyhedron knot made to date is a dodecahedron, made using a central jig and node jigs to hold basket cane in place before gluing. Planning was required to determine the cane's path between nodes (Nurse 2023). Figure 15 is based on a dodecahedron net, that is a set of pentagons that will fold in to a dodecahedron.

It is complex mapping dodecahedron edge configurations as per figure 11. Dodecahedrons have 30 edges, over a billion ($2^{30}$ or 1,073,741,824) edge configurations and up to 12 links. The equivalent icosahedron has the same number of edges and configurations but up to 20 sides. An algorithm to determine the number of links present in a given edge configuration may be possible.

I have made twin edge versions of all Platonic solids except the icosahedron. This shape has 5 edges meeting at each corner, a setup that is hard to make using current jigs.

**Polyhedra configurations**

Up till now, only one twin edged Platonic solid corner configuration has been examined. This section looks at other corner configurations.
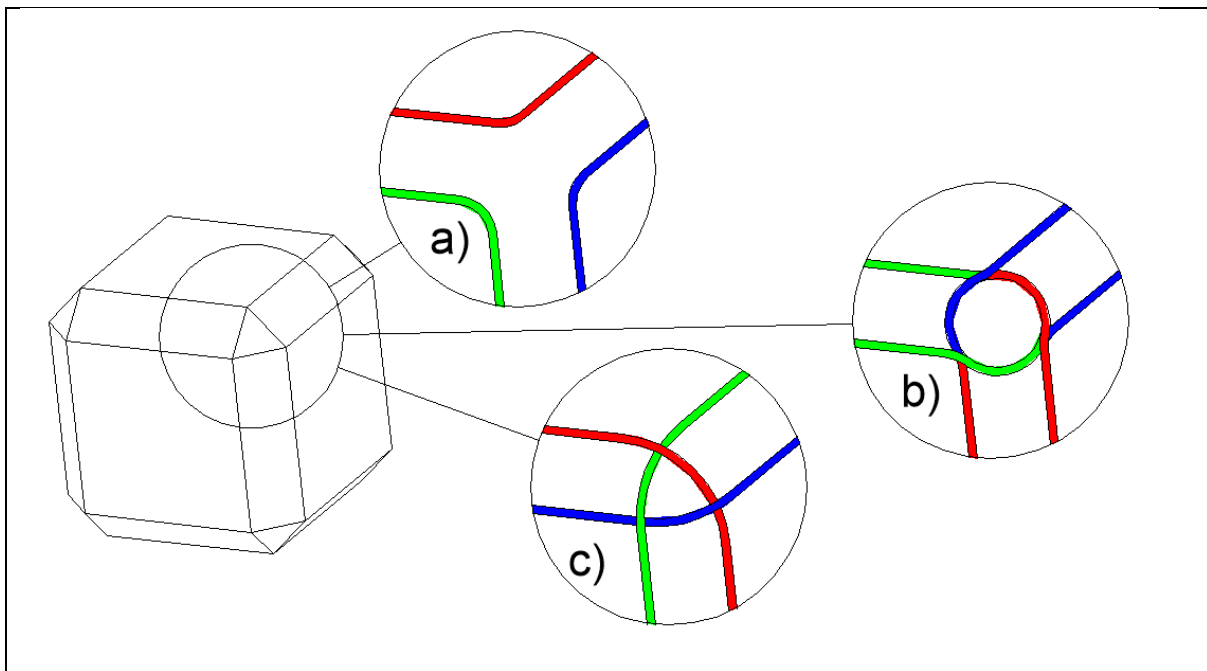


Figure 16 Twin edge cube with three possible corner styles

Looking at the figure 5 cube in detail, there are 3 node configurations preserving the twin edge arrangement. These are shown in figure 16: a) Edges don't cross and sides are independent without overlap or structure, b) links loop back on themselves and become edges, and c) overlapping links with structure as per figure 3. Note that 16c resembles figure 22 in Roelofs (2008), however Roelofs describes knots on each vertex of a polyhedron.

Figure 16 a), b) and c) have 240, 0 and 120 degree anticlockwise deviation angle of strands passing through the node. This applies when the node is flat, which occurs in hyperbolic (spherical) geometry.
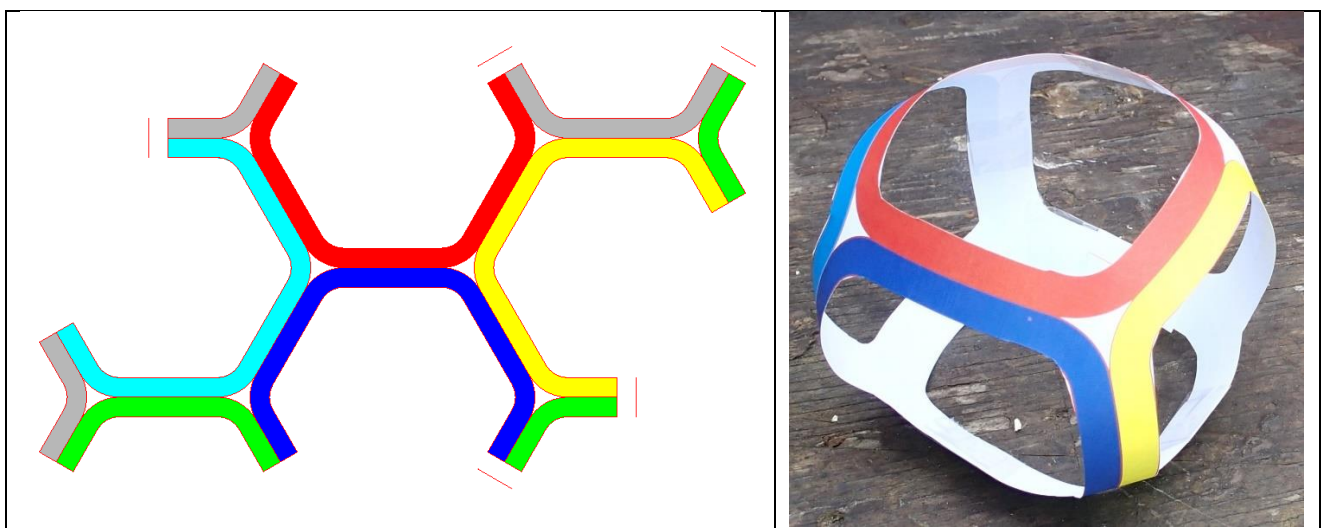


Figure 17: Edge net for cube 16a, with loops corresponding to sides.

Figure 18: Cube from figure 17 net.

Figure 19: Edge net for cube as per 16b.

Figure 20: Cube from figure 19 net.



Figure 21: Edge net for cube as per 16c, representing the topology of figures 1, 2 and 3.

Figure 22: Cube from figure 21 net.

These styles can be described by paper models. Models are made using 2d drawings with coloured in areas representing edges. Drawings are cut, rolled and taped. Figures 17 to 22 represent polyhedron styles shown in figure 16.

Figure 23: Sides or links are twisted together to form a cube. This is a structural configuration of figure 16a.

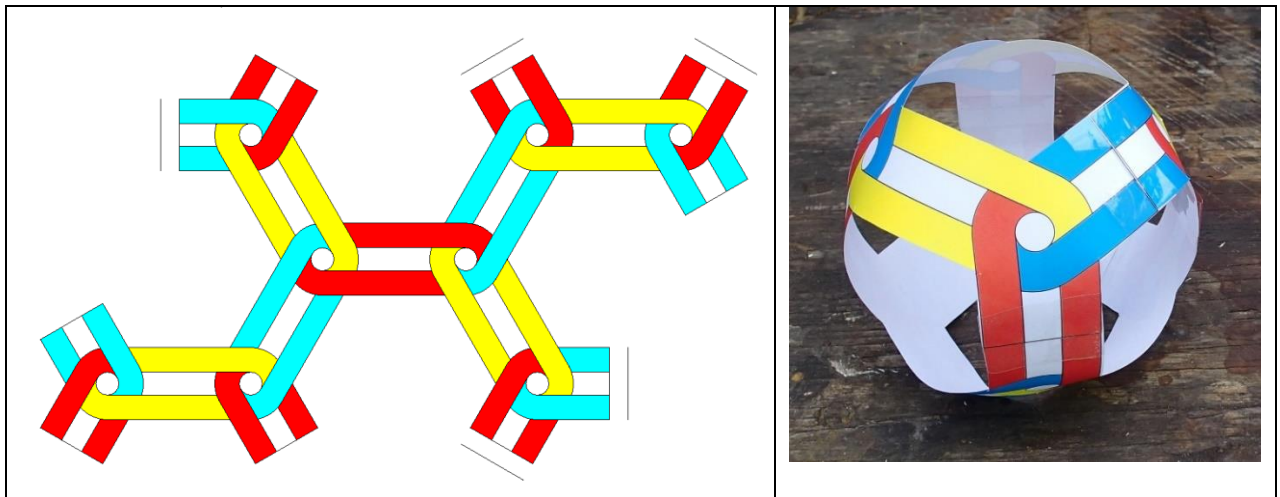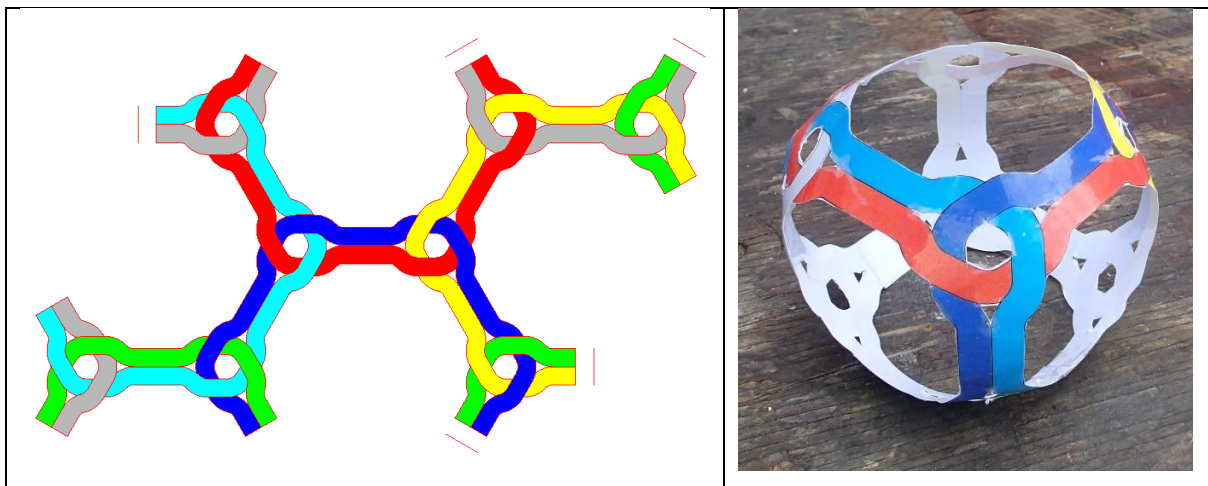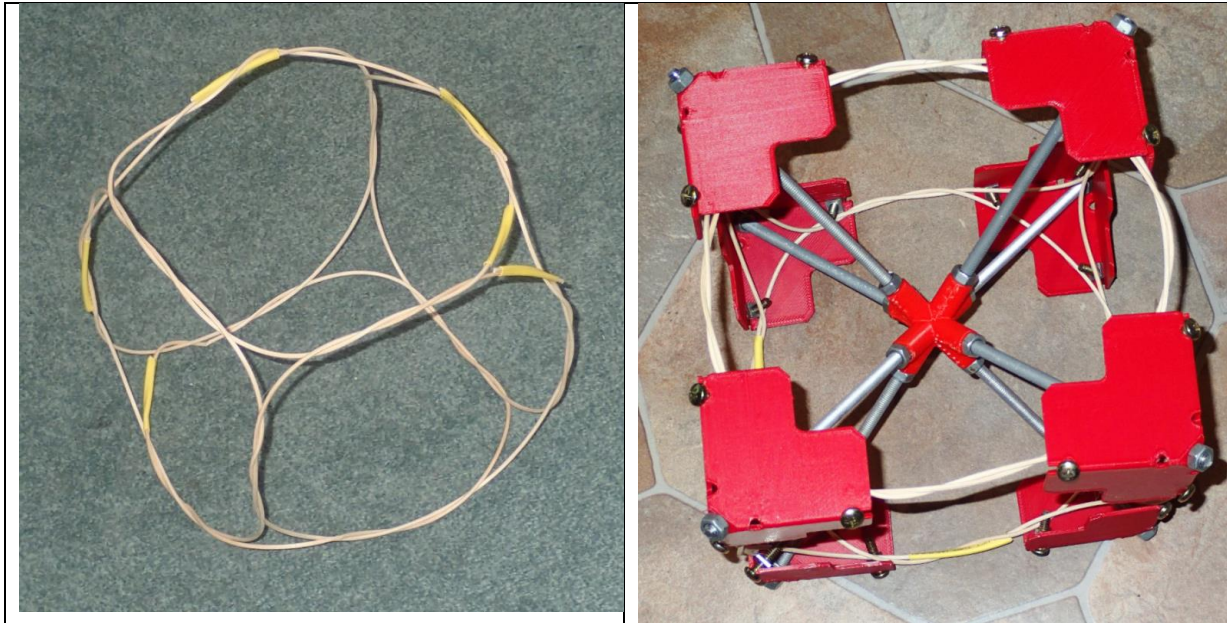Figure 24: Cube from figure 24 in construction jigs.

A configuration of a figure 16a style cube in shown in figure 24. Full twists along edges provide structure.



| Node Diagrams | Platonic Solid | Face x.1 | Edge x.2 | Parallel Face x.3 | Concave x.4 | x.5 |
|---|---|---|---|---|---|---|
| | Tetrahedron / 3 faces at corners | P1.1 | P1.2 | P1.3 | | |
| | Cube / 3 faces at corners | P2.1 | P2.2 | P2.3 | | |
| | Dodecahedron / 3 faces at corners | P3.1 | P3.2 | P3.3 | | |
| | Octahedron / 4 faces at corners | P4.1 | P4.2 | P4.3 | Skeletal octahedron P4.4 | |
| | Icosahedron / 5 faces at corners | P5.1 | P5.2 | P5.3 | Great dodecahedron P5.4 | Great dodecahedron P5.5 |

**DOUBLE-EDGED PLATONIC SOLIDS FROM LINKS**

Figure 25: Styles of twin edged Platonic solids with suggested nomenclature, (P3.2: P = Platonic Solid, 3 = Third solid, 2 = 2nd anti - clockwise deviation at node)
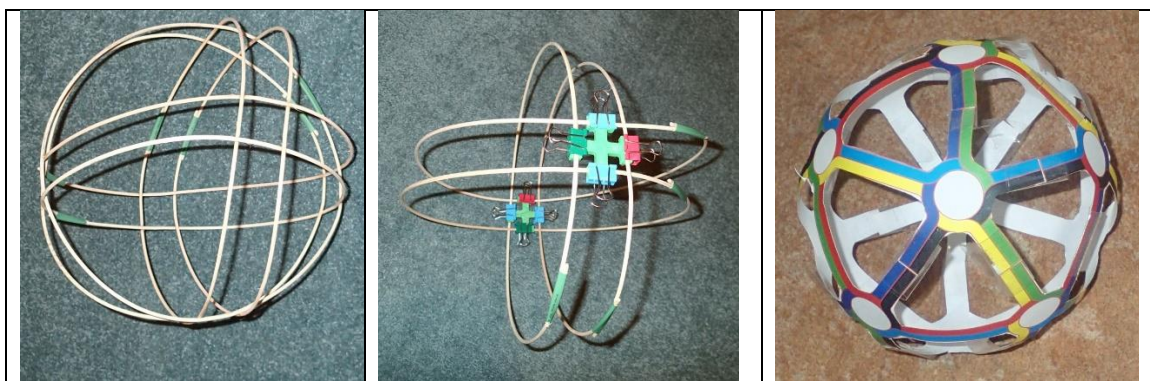
Figure 26: Concave skeletal octahedron (P4.4 in fig 20)

Figure 27: Skeletal octahedron in progress in jigs

Figure 28: Concave great dodecahedron (P5.5 in fig 20)

Styles of twin edge polyhedra from Platonic solids are tabled in figure 25. Styles P1.3, P2.1, P2.2, P2.3 have already been discussed. Tetrahedra, cubes and dodecahedra all have 3 faces at corners, but octahedra and icosahedra have 4 and 5 respectively. Rinus Roelofs (2008) makes similar statements, for example "There are five Platonic solids. Of these, the tetrahedron, the cube and the dodecahedron have in common the fact that at each vertex three faces meet."

Octahedral and icosahedral twin edge structures are shown in figures 26 to 28. They are unusual and include the zero volume skeletal octahedron. This octahedron is topologically the same as P2.3 seen in figure 3.

### Weaving Configurations

As well as the solid polyhedra listed in figure 25, planar polyhedra (tessellations from regular hexagons, squares and triangles) can be bases for twin-edged forms.
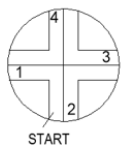
| DOUBLE-EDGED TILINGS / WEAVINGS FROM STRANDS AND LINKS | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Node Diagrams** | **Tessellation** | **x.1** | **x.2** | **x.3** | **x.4** | **x.5** | **x.6** |
| | | Face | Edge | Parallel Face | Other | | |
|  | **Hexagonal** 3 faces at corners | T1.1 | T1.2 | T1.3 | | | |
|  | **Square** 4 faces at corners | T2.1 | T2.2 | T2.3 | T2.4 | | |
|  | **Triangular** 6 faces at corners | T3.1 | T3.2 | T3.3 | T3.4 | T3.5 | T3.6 |

Figure 29: Styles of twin edged weavings based on tilings with suggested nomenclature, (T3.2: T = Tiling Based, 3 = Third tessellation, 2 = 2nd anti-clockwise deviation at nodes)

This is shown in figure 29, and examples of the weavings are shown in figures 30 and 31. In these representations, the black space is free or open. The diagrams show regular tilings which do not fill the plane.



Figure 30: Weaving T1.3, a hexagonal pattern where strands take the third diversion at nodes.
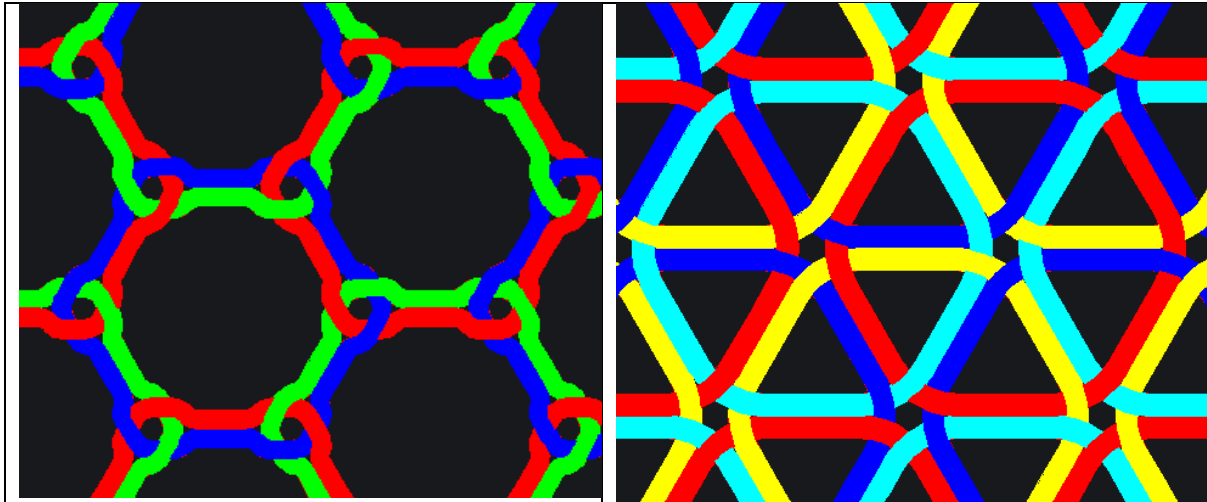
Figure 31: Weaving T3.6, a triangular and hexagon pattern where strands meeting at a node take the last diversion.

**Education and other applications**

There is considerable scope for demonstration and hands-on use of (doing) maths in the models shown in this paper. Topics include symmetry, hyperbolic geometry, indices, permutations, topology and network theory. The constructions could also be analysed using advanced mathematics, and appendix 1 is from an email from Daniel Mathews, a Monash University mathematician.

**Further Work on twin edged polyhedra and weavings**

More polyhedra could be made from basket cane including the remaining icosahedron, Archimedean solids, prisms, nanotube models from hexagons, antiprisms and antiprism stacks.

Mapping and counting links in configurations of double edge polyhedra could continue. One tetrahedron style has been mapped sofar. The number of possible configurations in 60 edged icosahedra and dodecahedra is large, and some automation or algorithm could be developed to predict the number of links in a given configuration.

The basic tessellation weavings could be modified to make continuous strand weavings capable of being stitched by machine.

**Summary**

Fresh work discussed in this research proposal came from curiosity about a friend's knitting pattern. First I made the cubic paper model as suggested in the pattern. Later, its topology was recognised, allowing variations to be made. The forms made correspond to the edges of chamfered / truncated Platonic solids as shown in figure 5. Altogether 18 styles of twin edge polyhedra and 13 weavings have been identified, each with possible configurations.  In some cases, configuration determines the number of links in a construction, for example in figures 6 to 10.

This fresh work could form the basis of curriculums and new maths. It could compliment similar work listed in the references (Nurse 2018, 2021, 2022)

**Dedication**

Dedicated to the memory of my wife Christine, who died in 2022. She made a crocheted cube which set the ball rolling.

**Appendix 1**

*Email between Steve Nurse and Daniel Mathews, Monash mathematics academic and researcher.*

From Daniel Mathews, 24/6/22

Hi Stephen,
I finally got around to reading your blogs. Nice stuff, and nice pictures. One way of looking at what you're doing is building links (i.e. knots with more than 1 component) out of polyhedra.

There is a sort of "standard" way in the literature to build links out of polyhedra (or more generally a planar graph), but what you're doing seems to be different.

This "standard" construction, I know as the "median construction". What you do is you put two strands with a twist (maybe half twist, maybe full twist, depending what you want to do) along each edge.

I found a nice illustration of this standard idea here (with half twist along each edge)
http://logical.ai/polyknot/org/celtic.html . I also mention it in this article
https://www.danielmathews.info/2017/09/08/tutte-meets-homfly/
(There's a theorem relating the Tutte polynomial of a graph with the HOMFLY polynomial of the link you get from the median construction... skip the algebra stuff if you want... some way down the page I mention the median constructions)
Also, in this paper with Tamas Kalman we also used the construction. It's about contact geometry and Floer homology but there are again pictures which illustrate the construction.
https://www.danielmathews.info/wp-content/uploads/2021/01/tight_contact_structures_on_seifert_surface_complements_published_version.pdf

But your construction is different from the median construction. I suppose it can be regarded as putting a pair of strands along each edge, but then something different "alternating" is happening... I don't think I've seen it before, but that's not to say it hasn't been considered before in the mathematical literature. As you mentioned, it has certainly been considered in the knitting literature!

Regards, Dan

**Acknowledgement**

Thanks to Daniel Mathews for his encouragement and advice.

**References**

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https://www.ravelry.com/patterns/library/gevlochten-bal---braided-ball

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https://www.thingiverse.com/thing:4741741/files

Penrose, R., 2005. *The road to reality: A complete guide to the laws of the universe*. Random house.

*Penrose describes maths which informs further analysis of the knots and links discussed in this proposal. Topics include hyperbolic geometry and knot theory.*

Roelofs, R., 2008, July. Connected holes. In *Bridges Leeuwarden: Mathematics, Music, Art, Architecture, Culture* (pp. 29-38). ( https://archive.bridgesmathart.org/2008/bridges2008-29.pdf )